Worcester County Mathematics League

Varsity Meet 2 - December 4, 2024

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League Varsity Meet 2- December 4, 2024





Round 1 - Fractions, Decimals, and Percents

1.
$$\frac{2890}{9}$$

2.
$$37.5 \text{ or } \frac{75}{2}$$

Round 2 - Algebra I

- 1. 412 or 412.00
- 2. 65
- 3. 23

Round 3 - Parallel Lines and Polygons

- 1. 165
- 2. 98
- 3. 33

Round 4 - Sequences and Series

- 1. 81
- 2. (5, 11, 17), exact order
- 3. 55,91 need both, either order

Round 5 - Matrices and Systems of Equations

$$1. \begin{bmatrix} 20 & -20 & 7 \\ -4 & 4 & 0 \end{bmatrix}$$

- 2. (5,3), (5,-3), (-5,3), (-5,-3) or $(\pm 5,\pm 3)$ need all four ordered pairs in any order
- 3. (-3, 3, -1), exact order

Team Round

1. 1000 or 1000.00

2.
$$\frac{p}{a - pq}$$
 or $-\frac{p}{pq - a}$

- 3. 70
- 4. 5
- 5. TMXRA, TAMXR, XRAMT and XRMTA (need all four permutations, listed in any order)
- 6. $0.05 \text{ or } \frac{1}{20}$

7.
$$\frac{\sqrt{5}-1}{2}$$
 or $\frac{\sqrt{5}}{2}-\frac{1}{2}$

- $8. \frac{1}{3}$
- 9. $\frac{65}{4}$

Worcester County Mathematics League Varsity Meet 2- December 4, 2024 Round 1 - Fractions, Decimals, and Percents



W. Boylston, Doherty, Mass Academy

All answers must be in simplest exact form in the answer section.

	NO CALCULATORS ALLOWED
1.	Convert the repeating decimal 321. $\overline{1}$ to a fraction in lowest terms m and n ; write your answer as $\frac{m}{n}$.
2.	When 140 is decreased by p percent, it is equal to 350 decreased by $2p$ percent. Find p.
3.	Laura read a book in three days. She read $\frac{1}{4}$ of the book and 20 more pages the first day. She read 0.375 of the remaining pages and 50 more pages the second day. On the third day, she read 60% of the remaining pages and then the last 20 pages to finish the book. How many pages in total did Laura read?
<u>AN</u>	NSWERS
(1 դ	pt) 1
(2 p	pts) 2. $p =$

(3 pts) 3. _____

Worcester County Mathematics League Varsity Meet 2- December 4, 2024 Round 2 - Algebra I



All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED
1. A team of five explorers agreed to be paid \$1236 per day, for the entire team. The leader of the team is paid twice the amount that is paid to each one of the other four explorers. How much does the leader make per day, in \$?
2. A train travels 208 miles between Boston and New York City. It makes 5 stops of 7 minutes each. It arrives in New York 3 hours and 47 minutes after departing Boston. What is the average speed of the train while in motion, in miles per hour?
3. A one foot wide grass border surrounds a rectangular pool with length l and width w . If l and w are both integer measures in feet, and the area of the pool equals the area of the border, find the sum of all possible values of w .
ANSWERS
(1 pt) 1. \$
$(2 \text{ pts}) \ 2. \ r =$

Blackstone Valley, St. John's/QSC, St. John's

(3 pts) 3. ____

Worcester County Mathematics League Varsity Meet 2- December 4, 2024 Round 3 - Parallel Lines and Polygons

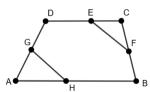


All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. What is the measure (in °) of an interior angle of a regular polygon with 24 sides?

2. Trapezoid ABCD is shown at right where E lies on \overline{CD} , F lies on \overline{BC} , H lies on \overline{AB} , and G lies on \overline{AD} . Given that $\overline{AB}||\overline{CD}, \overline{EF}||\overline{GH}, \text{ m}\angle A = 66^{\circ}$, and $\text{m}\angle CEF = 32^{\circ}$: find $\text{m}\angle DGH$.



3. A convex polygon with n sides has 65 fewer diagonals than a convex polygon with n+2 sides. Find n.

ANSWERS

- (1 pt) 1. _____°
- (2 pts) 2. ______
- (3 pts) 3. _____

Worcester County Mathematics League Varsity Meet 2- December 4, 2024 Round 4 - Sequences and Series



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Let x_n be the *n*-th term of a geometric sequence. If $x_3 = 9$ and $x_4 = 3$, find x_1 .

2. If the *n*-th sum of an arithmetic sequence is $S_n = 3n^2 + 2n$, find the first three terms (x_1, x_2, x_3) of the sequence, in that order.

3. Let $1 + 2 + 3 + ... + m = 1 + 2^2 + 3^2 + ... + n^2 = k$, where m > 1 and n > 1, find the two smallest values of k.

ANSWERS

(1 pt) 1. _____

(2 pts) 2.
$$(x_1, x_2, x_3) = \left(\frac{}{} \right)$$

(3 pts) 3. ______, _____

Worcester County Mathematics League Varsity Meet 2- December 4, 2024 Round 5 - Matrices and Systems of Equations



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find:

$$3\begin{bmatrix} 4 & -2 & -1 \\ 2 & 0 & 2 \end{bmatrix} - 2\begin{bmatrix} -4 & 7 & -5 \\ 5 & -2 & 3 \end{bmatrix}$$

2. Given the system of equations below, list all solutions as ordered pairs (x, y):

$$2x^2 - y^2 = 41$$
$$4x^2 + 3y^2 = 127$$

3. If $X = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ find ordered triple (a, b, c) such that:

$$X^{3} + aX^{2} + bX + cI = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where I is the 3 x 3 identity matrix.

ANSWERS

(2 pts) 2.
$$(x,y) \in \{ _ \}$$

(3 pts) 3.
$$(a, b, c) = ($$
_____)

Worcester County Mathematics League

Varsity Meet 2- December 4, 2024

Team Round



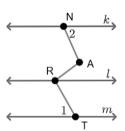
All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

- 1. Mrs. Dileo invested \$5000 in stocks and bonds. In the first year, her stock investment returned 14% and her bonds returned 6%. In total, her investments made a profit of \$620. How much of the \$5000 was invested in bonds?
- 2. Solve for r in terms of a, p, and q given that:

$$p = \frac{ar}{1 + qr}$$

3. Given k||l||m in the figure on the right, where m $\angle 1=60^\circ$, m $\angle A=100^\circ$, and $\overline{AR}\perp \overline{RT}$: find m $\angle 2$ in °.



4. Solve the following equation for n:

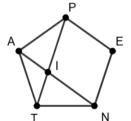
$$\sum_{k=1}^{n} 2^k = 62$$

5. Consider matrices M, A, T, R and X, where the dimensions (#rows x #columns) are given by:

$$M(3x2)$$
 $A(3x3)$ $T(2x3)$ $R(5x3)$ $X(2x5)$

List all possible ordered products of the five matrices, where each matrix appears exactly once, that produce a matrix with dimensions (2x3). For example, XMRTA and RMTAX are not ordered products that result in a (2x3) matrix.

6. If x is $\frac{1}{3}$ of 12.5% of 0.408, and x is p percent of 34, find p.



- 7. Given regular pentagon PENTA as shown at right, where PE=1 and diagonals \overline{AN} and \overline{PT} intersect in point I: find IT (the length of \overline{IT}).
- 8. The sequence below converges to L as the number of terms increases without bound. Find L.

$$\frac{1}{2}, \frac{2-1}{4}, \frac{4-(2-1)}{8}, \frac{8-(4-(2-1))}{16}, \frac{16-(8-(4-(2-1)))}{32}, \dots$$

9. Find the value of x if |A| is the determinant of matrix A and: $\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 14 & x \end{vmatrix}$.

Tantasqua, Worc. Acad. Tantasqua, Tantasqua, Doherty, Quaboag, AMSA, St. John's, Hudson

Worcester County Mathematics League Varsity Meet 2- December 4, 2024 Team Round Answer Sheet



ANSWERS



2.
$$r =$$

8.
$$L =$$

9.
$$x =$$

Worcester County Mathematics League Varsity Meet 2- December 4, 2024





Round 1 - Fractions, Decimals, and Percents

1.
$$\frac{2890}{9}$$

2.
$$37.5 \text{ or } \frac{75}{2}$$

Round 2 - Algebra I

- 1. 412 or 412.00
- 2. 65
- 3. 23

Round 3 - Parallel Lines and Polygons

- 1. 165
- 2. 98
- 3. 33

Round 4 - Sequences and Series

- 1. 81
- 2. (5, 11, 17), exact order
- 3. 55,91 need both, either order

Round 5 - Matrices and Systems of Equations

$$1. \begin{bmatrix} 20 & -20 & 7 \\ -4 & 4 & 0 \end{bmatrix}$$

- 2. (5,3), (5,-3), (-5,3), (-5,-3) or $(\pm 5,\pm 3)$ need all four ordered pairs in any order
- 3. (-3, 3, -1), exact order

Team Round

1. 1000 or 1000.00

2.
$$\frac{p}{a - pq}$$
 or $-\frac{p}{pq - a}$

- 3. 70
- 4. 5
- 5. TMXRA, TAMXR, XRAMT and XRMTA (need all four permutations, listed in any order)
- 6. $0.05 \text{ or } \frac{1}{20}$

7.
$$\frac{\sqrt{5}-1}{2}$$
 or $\frac{\sqrt{5}}{2}-\frac{1}{2}$

- $8. \frac{1}{3}$
- 9. $\frac{65}{4}$

Round 1 - Fractions, Decimals, and Percents

1. Convert the repeating decimal 321. $\overline{1}$ to a fraction in lowest terms m and n; write your answer as $\frac{m}{n}$.

Solution: Let $321.\overline{1}$. The straightforward method for converting repeating decimals to fractions starts by multiplying x by 10^k , where the ending of x, the repeating part, repeats every k digits. In this case, one digit (1) repeats, so k = 1. Then subtract x from $10^k \cdot x$, or 10x - x = 9x, in this case:

$$10x = 3211.111...$$

$$-x = 321.111...$$

$$9x = 2890.000... = 2890$$

and
$$x = \boxed{\frac{2890}{9}}$$

Alternative solution: Note that $321.\overline{1} = 321 + .\overline{1} = 321 + y$, where $y = .\overline{1} = \frac{1}{9}$. Then $x = 321 + \frac{1}{9} = \frac{9 \cdot 321 + 1}{9} = \frac{2889 + 1}{9} = \frac{2890}{9}$, as above.

2. When 140 is decreased by p percent, it is equal to 350 decreased by 2p percent. Find p.

Solution: Decreasing a number n by p percent is the same as decreasing n by the fraction $\frac{p}{100}$ of itself, which is the same as multiplying n by $1 - \frac{p}{100}$, or equivalently by $\frac{100-p}{100}$. So, 140 decreased by p% is equal to $140\frac{100-p}{100}$ and 350 decreased by 2p% is equal to $350\frac{100-2p}{100}$. Set these expressions equal and solve for p:

$$140\frac{100 - p}{100} = 350\frac{100 - 2p}{100}$$

$$\frac{140}{100} (100 - p) = \frac{350}{100} (100 - 2p)$$

$$\frac{7}{5} (100 - p) = \frac{7}{2} (100 - 2p)$$

$$2 (100 - p) = 5 (100 - 2p)$$

$$200 - 2p = 500 - 10p$$

$$10p - 2p = 8p = 500 - 200 = 300$$

and
$$8p = 300$$
, or $p = \frac{300}{8} = \frac{75}{2} = \boxed{37.5}\%$.

3. Laura read a book in three days. She read $\frac{1}{4}$ of the book and 20 more pages the first day. She read 0.375 of the remaining pages and 50 more pages the second day. On the third day, she read 60% of the remaining pages and then the last 20 pages to finish the book. How many pages in total did Laura read?

Solution: This problem is solved most easily by starting with the last information and working backwards.

Let b be the number of pages remaining to be read after the second day. On the third day, after reading 60% of b pages, there are 20 left, or $b\left(1-\frac{60}{100}\right)=20$, and $b\frac{40}{100}=20$, or $\frac{2b}{5}=20$, so that $b=\frac{5}{2}20=5\cdot 10=50$ pages.

Let a be the number of pages remaining to be read after the first day. On the second day, after reading $0.375 \cdot a$ pages, then another 50 pages, there are b = 50 pages left. Therefore a (1 - 0.375) = a(0.625) = 50 + 50 = 100. Note that $0.625 = \frac{5}{8}$, so $a\frac{5}{8} = 100$, and $a = \frac{8}{5}100 = 8\frac{100}{5} = 8 \cdot 20 = 160$.

Let n be the total number of pages in the book, that is, the number of pages that Laura read. Then $n\left(1-\frac{1}{4}\right)-20=a=160$, or $n\frac{3}{4}=160+20=180$, and $n=\frac{4}{3}180=4\frac{180}{3}=4\cdot60=\boxed{240}$ pages.

Round 2 - Algebra I

1. A team of five explorers agreed to be paid \$1236 per day, for the entire team. The leader of the team is paid twice the amount that is paid to each one of the other four explorers. How much does the leader make per day, in \$?

Solution: Let x be the daily pay for the four explorers. Then the leader is paid 2x, and 1236 = 2x + 4x = 6x. Thus, $x = \frac{\$1236}{6} = \206 , and the leader's pay is $2x = 2 \cdot \$216 = \frac{1236}{412}$.

Alternative solution: Each of the four explorers has a share in the team wage, and the supervisor has two shares. Thus there are 4+2=6 total shares and each share is $\frac{1}{6}$ of the team wage. Then the supervisor is paid two of the seven shares, $\frac{2}{6}=\frac{1}{3}$ of \$1236, which equals \$412.

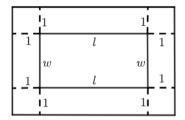
2. A train travels 208 miles between Boston and New York City. It makes 5 stops of 7 minutes each. It arrives in New York 3 hours and 47 minutes after departing Boston. What is the average speed of the train while in motion, in miles per hour?

Solution: First, convert the total travel time to minutes (1 hour = 60 minutes): 3 hours + 47 minutes = $3 \cdot 60 + 47 = 180 + 47 + 227$ minutes. Subtract the time when the train was stopped to calculate the time that the train was in motion: $227 - 5 \cdot 7 = 227 - 35 = 192$ minutes, or $\frac{192}{60}$ hours.

Let v be the average speed of the train, and recall that rate times time = distance travelled. Therefore $v\frac{192}{60}=208$ if v is measured in miles per hour. Solve for v: $v=208\cdot\frac{60}{192}=\frac{208\cdot5}{16}=13\cdot5=\frac{65}{16}$ miles per hour.

3. A one foot wide grass border surrounds a rectangular pool with length l and width w. If l and w are both integer measures in feet, and the area of the pool equals the area of the border, find the sum of all possible values of w.

Solution: Begin by drawing and labeling a figure such as one shown at right. The area of the pool is $l \cdot w$. The area of the border can be calculated by dividing it into two rectangles of dimension 1 X w, two rectangles of dimension l X 1, and four 1 X 1 squares, as shown in the figure at right. Thus the area of the border is 2w + 2l + 4.



Solution: (continued) Set the area of the pool equal to the area of the border: lw = 2w + 2l + 4. Subtract the area of the border from both sides so that all terms are on the left hand side of the equation, then factor:

$$lw - 2w - 2l - 4 = 0$$
$$(l-2)(w-2) - 4 - 4 = 0$$
$$(l-2)(w-2) = 8$$

Now l and w are integers, so l-2 and w-2 are also integers whose product is 8. The possible values of w-2 are then 1, 2, 4, and 8 (because 8 factors into either $1 \cdot 8$ or $2 \cdot 4$), and the possible values of w are 3, 4, 6, and 10. The sum of those values is $3+4+6+10=\boxed{23}$.

Round 3 - Parallel Lines and Polygons

1. What is the measure (in °) of an interior angle of a regular polygon with 24 sides?

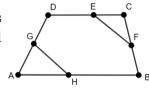
Solution: Let θ be the measure (in °) of a an interior angle of a regular polygon with n sides. Then θ can be calculated using the formula:

$$\theta = \frac{180(n-2)}{n}$$

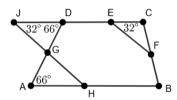
Substitute 24 for n and $\theta = \frac{180 \cdot 22}{36} = \frac{180}{24} \cdot 22 = \frac{15}{2} \cdot 22 = 15 \cdot 11 = \boxed{165}^{\circ}$.

Alternative solution: Recall that the exterior angles of any polygon sum to 360°. Also recall that the exterior angle and the interior angle at any vertex of a polygon are supplementary and that their measures sum to 180°. In this case, there are 24 exterior angles of equal measure, and a single exterior angle measures $\frac{360}{24} = 15^{\circ}$. Thus the measure of an interior angle is $180 - 15 = 165^{\circ}$.

2. Trapezoid ABCD is shown at right where E lies on \overline{CD} , F lies on \overline{BC} , H lies on \overline{AB} , and G lies on \overline{AD} . Given that $\overline{AB}||\overline{CD}$, $\overline{EF}||\overline{GH}$, $m\angle A=66^\circ$, and $m\angle CEF=32^\circ$: find $m\angle DGH$.



Solution: First, label the diagram with the given information. Next, extend segments \overline{HG} and \overline{CD} until they intersect at a point, labeled J on the diagram shown on the right. Note that \overline{HJ} and \overline{EF} are transversals of parallel lines \overline{AB} and \overline{CJ} . Likewise, \overline{AB} and \overline{CJ} are transversals of parallel lines \overline{HJ} and \overline{EF} .



Solution: (continued) Now $\angle CEF$ and $\angle DJG$ are corresponding angles (created by transversal \overline{CJ}) and are therefore congruent. More particularly, $m\angle DJG = m\angle CEF = 32^{\circ}$. Also, $\angle BAD$ and $\angle JDG$ are alternate interior angles (created by transversal \overline{AD} , and $m\angle JDG = m\angle A = 66^{\circ}$. Note than $\angle DGH$ and $\angle DGJ$ supplementary, so $m\angle DGJ = 180 - m\angle DGH$. Since the measures of the interior angles of a triangle sum to 180° , $m\angle DGH = 180 - (m\angle DJG + m\angle JDG) = 180 - (32 + 66) = 180 - 98 = 82^{\circ}$. Then $m\angle DGH = 180 - m\angle DGH = 180 - 82 = \boxed{98}^{\circ}$.

As a final note, the last few steps are simplified by noting that $\angle DGH$ is an exterior angle of $\triangle JDG$ and applying the exterior angle theorem (the measure of an exterior angle is equal to the sum of the two remote interior angles): $m\angle DGH = m\angle DJG + m\angle JDG = 32 + 66 = 98^{\circ}$.

3. A convex polygon with n sides has 65 fewer diagonals than a convex polygon with n+2 sides. Find n.

Solution: The number of diagonals in a convex polygon with n sides can be counted by first considering the number of idagonals connected to a given vertex. There are n vertices in a polygon with n sides, and a line segment can be drawn from any given vertex to each of the other n-1 vertices. Two of those n-1 line segments are sides of the polygon, and the other n-3 line segments are diagonals. Multiplying n vertices times n-3 diagonals per vertex overcounts the number of diagonals by a factor of 2, since each diagonal is counted twice, once by each of the two vertices that are endpoints of the diagonal. Thus, there are

$$\frac{n(n-3)}{2}$$

diagonals in an n sided convex polygon.

Likewise, there are $\frac{(n+2)(n-1)}{2}$ diagonals in a convex polygon with n+2 sides. Apply the given information and solve for n:

$$\frac{n(n-3)}{2} = \frac{(n+2)(n-1)}{2} - 65$$

$$\frac{n^2 - 3n}{2} = \frac{n^2 + n - 2}{2} - 65$$

$$n^2 - 3n = n^2 + n - 2 - 130$$

$$-3n = n - 132$$

$$132 = 4n$$

And
$$n = \frac{132}{4} = \boxed{33}$$
.

Round 4 - Sequences and Series

1. Let x_n be the *n*-th term of a geometric sequence. If $x_3 = 9$ and $x_4 = 3$, find x_1 .

Solution: Let $r = \frac{x_n}{x_{n-1}}$ be the common ratio of the geometric sequence. Then $r = \frac{x_4}{x_3} = \frac{3}{9} = \frac{1}{3}$. Recall that the *n*-th term of a geometric sequence is equal to $x_n = ar^{n-1}$, where $a = x_1$. Thus, $x_3 = x_1 \left(\frac{1}{3}\right)^{3-1} = x_1 \left(\frac{1}{3}\right)^2 = x_1 \left(\frac{1}{9}\right) = 9$, and $x_1 = 9 \cdot 9 = \boxed{81}$.

2. If the *n*-th sum of an arithmetic sequence is $S_n = 3n^2 + 2n$, find the first three terms (x_1, x_2, x_3) of the sequence, in that order.

Solution: Note that $S_1 = x_1$, $S_2 = x_1 + x_2$, and $S_3 = x_1 + x_2 + x_3$ and calcuate S_1 , S_2 , and S_3 :

$$S_1 = 3 \cdot 1^2 + 2 \cdot 1 = 3 + 2 = 5 = x_1$$

$$S_2 = 3 \cdot 2^2 + 2 \cdot 2 = 3 \cdot 4 + 4 = 12 + 4 = 16 = x_1 + x_2$$

$$S_3 = 3 \cdot 3^3 + 2 \cdot 3 = 3 \cdot 9 + 6 = 27 + 6 = 33 = x_1 + x_2 + x_3.$$

Then $S_2 - S_1 = x_1 + x_2 - x_1 = x_2 = 16 - 5 = 11$, and $S_3 - S_2 = x_1 + x_2 + x_3 - (x_1 + x_2) = x_3 = 33 - 16 = 17$. Collect all the results and $(x_1, x_2, x_3) = (5, 11, 17)$.

3. Let $1 + 2 + 3 + ... + m = 1 + 2^2 + 3^2 + ... + n^2 = k$, where m > 1 and n > 1, find the two smallest values of k.

Solution: Probably the quickest method to solve this problem is to simply list the terms of the two series and look for common terms. The first series consists of triangle numbers and may be familiar. If so, it makes sense to first calculate the terms of the second sequence (for n = 1, 2, 3, ...):

$$1, 5, 14, 30, 55, 91, 140, 204, 285, \dots$$

Them terms of the first series (for m = 1, 2, ...) are:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, \dots$$

Now recognize that the first three common terms are 1,55,91, and that the first common terms greater than 1 are 55,91.

Round 5 - Matrices and Systems of Equations

1. Find:

$$3\begin{bmatrix} 4 & -2 & -1 \\ 2 & 0 & 2 \end{bmatrix} - 2\begin{bmatrix} -4 & 7 & -5 \\ 5 & -2 & 3 \end{bmatrix}$$

Solution: — Let X be the matrix that is equal to the given expression. Recall that scalar multiplication, the multiplication of a matrix by a number (the scalar), is calculated by multiplying each member of the matrix by the scalar, in place. Therefore:

$$\mathbf{X} = \begin{bmatrix} 3 \cdot 4 & 3 \cdot (-2) & 3 \cdot (-1) \\ 3 \cdot 2 & 3 \cdot 0 & 3 \cdot 2 \end{bmatrix} + \begin{bmatrix} (-2) \cdot (-4) & (-2) \cdot 7 & (-2) \cdot (-5) \\ (-2) \cdot 5 & (-2) \cdot (-2) & (-2) \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -6 & -3 \\ 6 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 8 & -14 & 10 \\ -10 & 4 & -6 \end{bmatrix}$$

Next, recall that matrix addition of two matrices is calculated by adding the elements of the matrices "in place". Thus,

$$\mathbf{X} = \begin{bmatrix} 12+8 & (-6)+(-14) & (-3)+10 \\ 6+(-10) & 0+4 & 6+(-6) \end{bmatrix} = \begin{bmatrix} 20 & -20 & 7 \\ -4 & 4 & 0 \end{bmatrix}$$

2. Given the system of equations below, list all solutions as ordered pairs (x, y):

$$2x^2 - y^2 = 41$$
$$4x^2 + 3y^2 = 127$$

Solution: Solve this system by elimination after assigning new variables $u = x^2$ and $v = y^2$. Then subtract twice the first equation (2(2u - v = 41), or 4u - 2v = 82) from the second equation (4u + 3y = 127):

$$4u + 3v = 127$$

$$-4u + 2v = -82$$

$$-5v = 45$$

so that $v=y^2=9$. Substitute this value into the first equation and $2x^2-9=41$, or $2x^2=50$, or $x^2=25$ Now $x=\pm 5$ and $y=\pm 3$ may take on positive or negative values independently, resulting in four solutions: (5,3), (5,-3), (-5,3), (-5,-3)

3. If
$$X = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
 find ordered triple (a,b,c) such that:

$$X^{3} + aX^{2} + bX + cI = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where I is the 3 X 3 identity matrix.

Solution: Recall that the inner product definition of matrix multiplication states that for AB, the element in the *i*-th row and *j*-th column of the product matrix is equal to the dot product of the *i*th row of A and the *j*-th column of B. Then find X^2 , multiplying X by itself using this method:

$$X^{2} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, find X^3 , multiplying X^2 by X:

$$X^{2} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0 & 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 0 & 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 + 2 \cdot 0 & 0 \cdot 1 + 1 \cdot 1 + 2 \cdot 0 & 0 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting the matrices X^3 , X^2 , X, and I into the original matrix equation results in the equation of two (3 X 3) matrices with variables a, b, and c in the left hand matrix. Equating two (3 X 3) matrices generally results in 9 equations, one for each element of the matrix. In this case, all the matrices are upper triangle matrices, and three of the equations are simply 0 = 0. In addition, all of the four matrices are constant along the main diagonal and symmetric across the main cross diagonal, so that three of the remaining equations are redundant.

Solution: (continued) Therefore write the three equations in three unknowns a, b, and c that result from summing the first row from each of the four matrices:

$$1 + a \cdot 1 + b \cdot 1 + c \cdot 1 = 1 + a + b + c = 0$$
$$3 + a \cdot 2 + b \cdot 1 + c \cdot 0 = 3 + 2a + b = 0$$
$$6 + a \cdot 3 + b \cdot 1 + c \cdot 0 = 6 + 3a + b = 0$$

Subtract the second equation from the first equation and 3+a=0, or a=-3. Substitute a=-3 into the second equation and 3+2(-3)+b=3-6+b=-3+b=0, and b=3. Finally, substitute a=-3 and b=3 into the first equation and 1+(-3)+3+c=1+c=0, or c=-1. Collect these results and $(a,b,c)= \overline{(-3,3,-1)}$.

Note that X^2 and X^3 might be more simply calculated given familiarity with the outer product definition of matrix multiplication. Thus,

$$X^{2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and $X^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, agreeing with the result that was calculated using the inner product.

As a final note, the Cayley-Hamilton theorem says that every matrix solves its own characteristic equation. The characteristic equation for X is $p(\lambda) = \det(\lambda I - X) = 0$. This determinant is simple to find because the matrix $\lambda I - X$ is upper triangular, and the determinant of a triangular matrix is the product of the elements on its main diagonal. In this case, the main diagonal elements all equal $\lambda - 1$, so that $p(\lambda) = (\lambda - 1)^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = \lambda^3 + a\lambda^2 + b\lambda + c$, and (a, b, c) = (-3, 3, -1).

Team Round

1. Mrs. Dileo invested \$5000 in stocks and bonds. In the first year, her stock investment returned 14% and her bonds returned 6%. In total, her investments made a profit of \$620. How much of the \$5000 was invested in bonds?

Solution: Let *b* be the amount, in \$, that Mrs. Dileo invested in bonds. Then 5000-b is the amount she invested in stocks. She earned $b \frac{6}{100} = \frac{3b}{50}$ (in \$) on bonds and $(5000-b) \frac{14}{100} = 5000 \frac{14}{100} - b \frac{14}{100} = 50 \cdot 14 - \frac{7b}{50} = 700 - \frac{7b}{50}$ on bonds. The sum of those two expressions is equal to \$620, Mrs. Dileo's profit. Then $\frac{3b}{50} + 700 - \frac{7b}{50} = \$700 - \frac{4b}{50} = \$620$. Solve for *b*, and $\frac{4b}{50} = \frac{2b}{25} = 700 - 620 = \80 , or $b = 80\frac{25}{2} = 40 \cdot 25 = \$ \boxed{1000}$.

2. Solve for r in terms of a, p, and q given that:

$$p = \frac{ar}{1 + qr}$$

Solution: First multiply both sides of the equation by the denominator in the rational expression on the right side. Then distribute multiplication over addition, isolate the terms with r on one side of the equation, factor out r, and finally divide the expressions on both sides of the equation by the factor that is multiplied by r:

$$p = \frac{ar}{1+qr}$$

$$p(1+qr) = ar$$

$$p+pqr = ar$$

$$pqr = ar - p$$

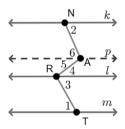
$$pqr - ar = -p$$

$$(pq - a)r = -p$$

$$r = \frac{-p}{pq - a} = \boxed{\frac{p}{a-pq}}$$

3. Given k||l||m as shown in the figure below, where $m\angle 1=60^\circ$, $m\angle A=100^\circ$, and $\overline{AR}\perp \overline{RT}$: find $m\angle 2$. Note: drawings are not to scale.

Solution: First draw line p through A and parallel to k, l, and m, shown as a dotted line in the figure to the right. Note that line l divides $\angle TRA$ into two angles; label these $\angle 3$ and $\angle 4$, as shown. Note also that line p divides $\angle A$ into two angles; label these $\angle 5$ and $\angle 6$, as shown. Next, \overline{TR} transverses parallel line l and m so that opposite interior angles 1 and 3 have the same measure, and $m\angle 3 = m\angle 1 = 60^{\circ}$.



Solution: (continued) Since $\overline{AR} \perp \overline{RT}$, $m\angle 3 + m\angle 4 = m\angle TRA = 90^\circ$, and $m\angle 4 = 90 - m\angle 3 = 90 - 60 = 30^\circ$. Now \overline{RA} transverses parallel lines l and p and $\angle 4$ and $\angle 5$ are alternate interior angles, so that $m\angle 5 = m\angle 4 = 30^\circ$. Proceeding upwards, $m\angle A = 100^\circ = m\angle 5 + m\angle 6$, so that $m\angle 6 = 100 - m\angle 5 = 100 - 30 = 70^\circ$. Finally, \overline{AN} traverses parallel lines p and k, with alternate interior angles $m\angle 6$ and $m\angle 2$. Therefore $m\angle 2 = m\angle 6 = \boxed{70}^\circ$.

4. Solve the following equation for n:

$$\sum_{k=1}^{n} 2^k = 62$$

Solution: Rewrite the summation as:

$$2 + 2^2 + 2^3 + \dots + 2^n = 2 + 4 + 8 + \dots + 2^n = 62$$

Note that $2^6 = 64 > 62$, so that n = 6 is too large. Then n = 5 is a reasonable guess. Check that the sum of the exponents of 2 from 2^1 to $2^5 = 32$ is equal to 62. It does, and $n = \boxed{5}$.

5. Consider matrices M, A, T, R and X, where the dimensions (#rows x #columns) are given by:

$$M(3x2)$$
 $A(3x3)$ $T(2x3)$ $R(5x3)$ $X(2x5)$

List all possible ordered products of the five matrices, where each matrix appears exactly once, that produce a matrix with dimensions (2x3). For example, XMRTA and RMTAX are not ordered products that result in a (2x3) matrix.

Solution: First, given a legal matrix product PQ, the number of columns in P must equal the number of rows in Q. Moreover, if P has dimensions (mxn) and Q has dimensions (pxq), then the product PQ will have dimensions (mxq). Thus, for a product of the five matrices to have dimensions (2x3), the first, leftmost matrix in the product must have 2 rows and the last, rightmost matrix in the product must have 3 columns. There are two possibilities for the leftmost matrix: T and X. There are three possible rightmost matrices: A, T and R.

Note that the product XR has dimensions (2x3) and must appear in that order in any legal product of the five matrices; that is because X is the only matrix with 5 columns and R is the only matrix with 5 rows. If X is the first, leftmost matrix, then there are two possible products: XRMTA and XRAMT. If not, then T is the leftmost matrix, M must immediately preced XR in the product, and there are two more possible products: TMXRA and TAMXR, and these four products are the only possible orderings that meet the conditions: XRMTA, XRAMT, TMXRA, TAMXR.

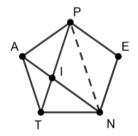
6. If x is $\frac{1}{3}$ of 12.5% of 0.408, and x is p percent of 34, find p.

Solution: First, note that 12.5% is $\frac{1}{8}$ expressed as a fraction. Therefore $\frac{1}{3}$ of 12.5% is $\frac{1}{3}$ of $\frac{1}{8}$, or $\frac{1}{3} \cdot \frac{1}{8} = \frac{1}{24}$, and $\frac{1}{24}$ of 0.408 equals $0.408 \div 24 = (0.48 - 0.072) \div 24 = 0.48 \div 24 - 0.072 \div 24 = 0.02 - 0.003 = 0.017 = x$. Therefore x is $100 \cdot (0.017 \div 34)$ percent of 34. Thus, $p = \frac{100 \cdot 0.017}{34} = \frac{1}{10} \cdot \frac{17}{34} = \frac{1}{10} \cdot \frac{17}{34} = \frac{1}{10} \cdot \frac{1}{2} = \boxed{\frac{1}{20}}$.

7. Given regular pentagon PENTA as shown below right, where PE=1 and diagonals \overline{AN} and \overline{PT} intersect in point I: find IT (the length of \overline{IT}).

Solution: First, construct diagonal \overline{PN} . This construction will be used to prove that $\triangle PTN$ and $\triangle NIT$ are similar isosceles triangles. For now, accept this fact, as well as the fact that $\triangle NIT$ and $\triangle PAI$ are congruent.

Next, let IT = x. PENTA is a regular polygon with congruent sides and TN = PE = 1. Also, IN = TN = 1 = IP because $\triangle NIT$ is isosceles and congruent to $\triangle PAI$.



Solution: (continued) Write the proportion $\frac{PT}{NI} = \frac{TN}{IT}$ (because $\triangle PTN \sim \triangle NIT$). Note that PT = IP + IT = 1 + x and replace NI = TN = 1 and IT = X:

$$\frac{1+x}{1} = \frac{1}{x}$$

Multiply both sides of the equation by x and subtract 1 from both sides, resulting in:

$$x(1+x) - 1 = x^2 + x - 1 = 0$$

Now use the quadratic formula to solve for x, and $x = \frac{-1 \pm \sqrt{1^2 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$. Discard the negative solution and $x = IT = \boxed{\frac{\sqrt{5} - 1}{2}}$.

Proof. The similarity of $\triangle PTN$ and $\triangle NIT$ can be proved using Lemma 1 and Lemma 2, which are stated and proved below. In particular, $m\angle ITN = m\angle PNT = 72^{\circ}$ by Lemma 2. Also, $m\angle TNI = 36^{\circ}$ by Lemma 1. Then $m\angle NIT = 180 - 36 - 72 = 72^{\circ}$ because the measures of the angles of $\triangle NIT$ sum to 180° . Thus, $\triangle PTN$ and $\triangle NIT$ each have two angles that measure 72° , and $\triangle PTN \sim \triangle NIT$ by AA Similarity. Finally, $m\angle API = 36^{\circ}$ by Lemma 1, and $m\angle PAI = 72^{\circ}$ by Lemma 2 and the fact that it is a base angle of $\triangle NPA$, and $\overline{AP} \cong \overline{TN}$, so $\triangle NIT \cong \triangle PAI$. \square

Lemma 1. If an isosceles triangle is formed by a diagonal and two adjacent sides of a regular pentagon, its base angles measure 36°.

Proof. Note that the vertex angle of the triangle is an internal angle of the pentagon. Its measure is therefore 108° : the sum of the internal angles of a polygon with n sides is $(n-2)180^{\circ}$, or $3 \cdot 180 = 540^{\circ}$ for a pentagon; and the five angles of the pentagon have equal measure, that is, $540 \div 5 = 108^{\circ}$. If the measure of a base angle is x, then 2x + 108 = 180 because the sum of the measure of the angles in a triangle is 180° , $2x = 180 - 108 = 72^{\circ}$, and $x = 36^{\circ}$.

Solution: (continued)

Lemma 2. If a triangle is formed by two diagonals and one side of a regular pentagon, its base angles measure 72°.

Proof. Note that each diagonal divides a 108° internal angle of the pentagon into two angles. One of those angles is a base angle of a triangle formed by two pentagon sides and the diagonal in question. The other angle is a base angle of the triangle with two diagonals as sides. The angle measure of the first angle is 36° by Lemma 1. The measure of the second angle is $108-36=72^{\circ}$. \square

8. The sequence below converges to L as the number of terms increases without bound. Find L.

$$\frac{1}{2}, \frac{2-1}{4}, \frac{4-(2-1)}{8}, \frac{8-(4-(2-1))}{16}, \frac{16-(8-(4-(2-1)))}{32}, \dots$$

Solution: This problem is most quickly solved by observing that the given sequence is actually a geometric series (in disguise!). To see this fact, let the terms of the sequence be $s_1, s_2, s_3, ...$ and note that $s_1 = 2^{-1}$, $s_2 = \frac{2^1 - 2^0}{2^2} = 2^{-1} - 2^{-2}$, $s_3 = \frac{2^2 - (2^1 - 2^0)}{2^3} = \frac{2^2 - 2^1 + 2^0}{s^3} = 2^{-1} - 2^{-2} + 2^{-3}$, and

$$s_k = \frac{2^{k-1} - (2^{k-2} - (2^{k-3} - (\dots)))}{s^k} = \frac{2^{k-1} - 2^{k-2} + 2^{k-3} - 2^{k-4} + \dots}{2^k} = 2^{-1} - 2^{-2} + 2^{-3} - 2^{-4} + \dots$$

in general. The sequence is therefore an infinite geometric series with ratio $r=-\frac{1}{2}$ and initial value $a=\frac{1}{2}$. Recall the formula for an infinite geometric series: $S=\frac{a}{1-r}$. Then $L=\frac{\frac{1}{2}}{1-(-\frac{1}{2})}=\frac{1}{2}$

$$\frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{2} \cdot \frac{2}{3} = \boxed{\frac{1}{3}}.$$

9. Find the value of x if |A| is the determinant of matrix A and: $\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 14 & x \end{vmatrix}$.

Solution: Recall that the determinant of 2 X 2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is equal to ad - bc. Then $\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} = 2 \cdot 1 - 3 \cdot 7 = 2 - 21 = -19$ and $\begin{vmatrix} 4 & 6 \\ 14 & x \end{vmatrix} = 4x - 6 \cdot 14 = 4x - 84$.

Set these two expressions equal: -19 = 4x - 84. Then 4x = 84 - 19 = 65, and $x = \left\lfloor \frac{65}{4} \right\rfloor$